ONE-DIMENSIONAL PROBLEM OF THE SOLUTION AND LEACHING OF SALTS AT HIGH VALUES OF THE PECLET NUMBER

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 2, pp. 148–152, 1969

If the filtration rate of a solution is much greater than the diffusion rate of the dissolved substance, the one-dimensional problem of the solution and leaching of salts in soils reduces to the integration of a system of equations of the form [1]

$$vC_x + \sigma C_t - a_1 \sigma N^\alpha (C_* - C) = 0$$
$$N_t + a_1 N^\alpha (C_* - C) = 0. \qquad (0.1)$$

Here, v = const is the filtration rate, σ is the porosity, $\alpha \geq 0$, $a_1 > 0$ are constants depending on the salinity of the soil, $C = C(x, t)$ is the concentration of the solution, $N = N(x, t)$ is the content of salts in the solid phase per unit volume of soil, C_* is the limiting saturation concentration, x is a coordinate, and t is time.

Given certain assumptions, problems of suspension flow in a porous medium accompanied by "colmatage-suffosion" effects [2] reduce to equations analogous to system (0.1).

Equations (0.1) constitute a quasi-linear (for $\alpha \neq 0$) system of hyperbolic type with two families of characteristics

$$x = \text{const}, \quad x - vt / \sigma = \text{const}.$$

Transforming to dimensionless quantities

$$c = C / C_*, \quad n = N / C_*, \qquad a = C_*^\alpha \sigma l a_1 / v$$

where l is a certain characteristic linear dimension, and substituting for the independent variables the notation

$$x_1 = x / l, \quad x_2 = (x - vt / \sigma) / l \qquad (0.2)$$

we write (0.1) in the form:

$$\frac{\partial c}{\partial x_1} - \frac{\partial n}{\partial x_2} = 0, \qquad \frac{\partial n}{\partial x_2} = a(1 - c) n^\alpha. \qquad (0.3)$$

We note that in [1] the solutions of certain problems are presented for system (0.1) with $\alpha = 0, 0.5$ on the assumption that the quantity σc_t can be neglected. Below, we examine the analogous problems without this constraint, leaving aside the somewhat trivial case $\alpha = 0$.

1. Infiltration of fresh water into dry saline soil. In this case it is required to solve system (0.3) in the region $x_1 > 0$, $x_2 < 0$ with the conditions

$$c = 0, \quad x_1 = 0, \; x_2 \leqslant 0; \qquad n = F(x_1), \quad x_1 \geqslant 0, \; x_2 = 0. \qquad (1.1)$$

We assume that $F(x_1)$ is a differentiable function.

As will be clear from what follows, the solutions of problem (0.3)–(1.1) will be qualitatively different depending on whether $\alpha < 1$ or $\alpha \geq 1$.

a) We assume that $\alpha = 1$. Determining the function c from the second equation of system (0.3)

$$c = 1 - \frac{1}{a} \frac{\partial \ln n}{\partial x_2} \qquad (1.2)$$

and substituting it into the first equation, we find

$$\frac{1}{a} \frac{\partial^2 \ln n}{\partial x_2 \partial x_1} + \frac{\partial n}{\partial x_2} = 0.$$

Integrating this with respect to x_2, we obtain

$$\frac{1}{a}\frac{\partial \ln n}{\partial x_1} + n = f_1(x_1) \tag{1.3}$$

where $f_1(x_1)$ is an arbitrary function.

It is easy to see that the second of conditions (1.1) gives

$$f_1(x_1) = \frac{1}{a}\frac{\partial \ln F(x_1)}{\partial x_1} + F(x_1).$$

Substituting this expression in (1.3) and integrating the equation obtained, we find

$$\frac{n}{F - n} = f_2(x_2)\exp\left\{a\int_0^{x_1} F\,dx\right\}. \tag{1.4}$$

Here, $f_2(x_2)$ is an arbitrary function.

Setting $x_1 = 0$ in (1.2), using conditions (1.1) and integrating the expression obtained, we find

$$n|_{x_1=0} = F(0)\exp(ax_2).$$

This makes it possible to determine the function f_2 from (1.4):

$$f_2 = [1 - \exp(ax_2)]^{-1}\exp(ax_2)$$

and hence the unknown function $n(x_1, x_2)$

$$n(x_1, x_2) = F(x_1)\exp\left(a\int_0^{x_1} F\,dx\right)\left[\exp(-ax_2) + \exp\left(a\int_0^{x_1} F\,dx\right) - 1\right]^{-1}. \tag{1.5}$$

We find the instantaneous distribution of the solution concentration from Eq. (1.2):

$$c = \left[\exp\left(a\int_0^{x_1} F\,dx\right) - 1\right]\left[\exp(-ax_2) + \exp\left(a\int_0^{x_1} F\,dx\right) - 1\right]^{-1}. \tag{1.6}$$

Substituting (0.2) into (1.5) and (1.6) and going over to dimensional quantities completes the solution of the initial physical problem.

In Fig. 1 the solid and dashed lines represent graphs of relations (1.5) and (1.6), respectively, for the case

$$a = 1, \qquad \tau = \frac{vt}{l\sigma} = 1, 2, 3, \qquad F(x_1) = 5\exp(-4x_1).$$

b) We now consider the case $\alpha < 1$. To simplify the calculations we set $\alpha = 0.5$; $F(x_1) = n_0 = \text{const}$.

Proceeding as before, we have

$$\partial n / \partial x_1 = an^{1/2}(n_0 - n).$$

Separating variables and integrating, we find

$$(n_0^{1/2} + n^{1/2})(n_0^{1/2} - n^{1/2})^{-1} = f_3(x_2)\exp(ax_1 n_0^{1/2}). \tag{1.7}$$

For $x_1 = 0$, $c = 0$ from the second equation of system (0.3) we obtain

$$\left.\frac{\partial n}{\partial x_2}\right|_{x_1=0} = an^{1/2}|_{x_1=0}.$$

After integration with allowance for the boundary condition we obtain

$$n^{1/2}|_{x_1=0} = {}^1\!/_2 ax_2 + n_0^{1/2}. \tag{1.8}$$

Setting $x_1 = 0$ in (1.7) and comparing with (1.8), we determine the function f_3:

$$f_3 = \frac{2}{ax_2}\left(2n_0^{1/2} + \frac{ax_2}{2}\right)$$

and hence rewrite Eq. (1.7) in the form:

$$n^{1/2} = n_0^{1/2}\frac{n_0^{1/2} + ax_2[1 + \exp(-an_0^{1/2}x_1)]/4}{n_0^{1/2} + ax_2[1 - \exp(-an_0^{1/2}x_1)]/4} \tag{1.9}$$

while the function c is found from the second equation of system (0.3):

$$c(x_1, x_2) = 1 - n_0 \exp(-a n_0^{1/2} x_1) \{n_0^{1/2} + a x_2 [1 - \exp(-a n_0^{1/2} x_1)] / 4\}^{-2}. \tag{1.10}$$

It is easy to see that

$$n(0, -2n_0^{1/2} / a) = 0$$

and system (0.3) degenerates. We assume that there exists a line $x_2 = F(x_1)$ passing through the point $(0, -2n_0^{1/2}/a)$ such that

$$n(x_1, x_2) = c(x_1, x_2) = 0, \qquad (x_2 = F(x_1)). \tag{1.11}$$

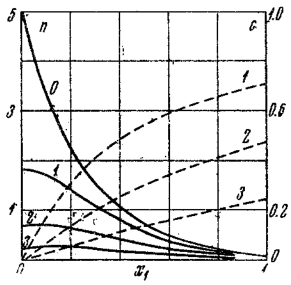

Fig. 1

Denoting by n^+, c^+ the functions determined by Eqs. (1.9) and (1.10), respectively, and by c^-, n^- the unknown functions defined in the region bounded by the characteristic $x_2 = -2n_0^{1/2}/a$ and a line $x_2 = F(x_1)$, in accordance with expression (1.7) we have

$$(n^-)^{1/2} = n_0^{1/2} [f_4(x_2) - \exp(-a n_0^{1/2} x_1)] [f_4(x_2) + \exp(-a n_0^{1/2} x_1)]^{-1}. \tag{1.12}$$

Requiring satisfaction of the condition of continuity of the function n on the above-mentioned characteristic, we obtain

$$f_4(-2n_0^{1/2} / a) = f_3(-2n_0^{1/2} / a) = 1. \tag{1.13}$$

Substituting (1.12) into the second equation of system (0.3), we find

$$c = 1 - \frac{4n_0^{1/2} f_4' \exp(-a n_0^{1/2} x_1)}{a [f_4 + \exp(-a n_0^{1/2} x_1)]^2} \qquad \left(f_4' = \frac{df_4}{dx_2}\right).$$

From conditions (1.11) written for n^-, c^- it follows that the equation of the unknown line can be written in the form:

$$x_1 = -a^{-1} n_0^{-1/2} \ln f_4$$

where the function $f_4(x_2)$ satisfies the differential equation

$$f_4' / f_4 = a n_0^{-1/2}.$$

Integrating this equation with allowance for initial condition (1.13), we find

$$f_4 = \exp(2 + a x_2 n_0^{-1/2}).$$

Thus, the required line is the straight line given by the equation

$$x_1 = -n_0^{-1/2} (2 + a x_2 n_0^{-1/2}) / a$$

and the final solution of the problem is written in the form:

$$c(x_1, x_2);\ n(x_1, x_2) = \begin{cases} c^+, \quad n^+, \quad -2n_0^{1/2} / a \leqslant x_2 \leqslant 0, \quad x_1 > 0 \\ c^-, \quad n^-, \quad -(2n_0^{1/2} / a + n_0 x_1) < x_2 \leqslant -2n_0^{1/2} / a, \quad x_1 > 0 \\ 0, \quad 0, \quad x_2 \leqslant -(2n_0^{1/2} / a + n_0 x_1), \quad x_1 > 0. \end{cases}$$

From these solutions it is clear that in the case $\alpha < 1$ starting from a certain time an expanding salt-free zone appears. When $\alpha \geq 1$ there is no such zone, although the solid-phase salt content tends to zero with time.

2. Infiltration into a soil containing partially dissolved salts. At $t = 0$ let the pores of the soil sample contain a salt solution of concentration $c_0 = \text{const} < 1$ and salt in the solid phase with a distribution density $n_0 = \text{const} > 0$, and at

$t > 0$ let fresh water begin to be supplied at a constant rate at the end $x_1 = 0$. Moreover, let the type of salinity correspond to the case $\alpha = 1$. In the plane of the variables x_1x_2, introduced by relations (0.2), the problem is formulated as follows: find the functions (x_1, x_2) and $n(x_1, x_2)$, satisfying the system

$$\frac{\partial c}{\partial x_1} = \frac{\partial n}{\partial x_2}, \qquad \frac{\partial n}{\partial x_2} = an(1-c) \tag{2.1}$$

in the region $x_1 > 0$, $x_2 < x_1$, apart from the line $x_2 = 0$, and the conditions

$$c = c_0, \quad n = n_0, \quad x_1 = x_2 \quad (x_1 > 0)$$
$$c = 0, \quad x_1 = 0, \quad x_2 \leqslant 0. \tag{2.2}$$

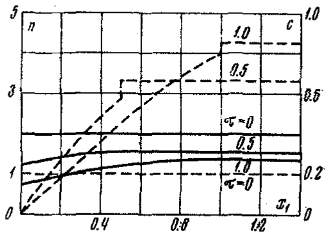

Fig. 2

The function n is required to be continuous throughout the region of definition. As follows from conditions (2.2) and the hyperbolicity of system (2.1), the function c will have a discontinuity of the first kind along the characteristic $x_2 = 0$.

We denote by c^+, n^+ and c^-, n^- functions satisfying system (2.1) and defined in the regions D_1: $\{0 < x_2 < x_1\}$ and D_2: $\{x_1 > 0, x_2 < 0\}$ respectively.

In accordance with (1.2) and (1.3) we have

$$\frac{1}{a}\frac{\partial \ln n^{\pm}}{\partial x_1} + n^{\pm} = f_1^{\pm}(x_1), \qquad c^{\pm} = 1 - \frac{1}{a}\frac{\partial \ln n^{\pm}}{\partial x_2}. \tag{2.3}$$

In order to determine f_1^+ we differentiate along the straight line $x_2 = x_1$ the equation $n^+|_{x_1=x_2} = n$

$$\left[\frac{\partial n^+}{\partial x_1} + \frac{\partial n^+}{\partial x_2}\right]_{x_1=x_2} = 0.$$

Substituting into (2.3) and keeping in mind conditions (2.2), we find

$$f_1^+ = c_0 + n_0 - 1.$$

Let $c_0 + n_0 - 1 \neq 0$. Separating the variables and integrating, from the first equation of system (2.3) we obtain

$$n^+(c_0 + n_0 - 1 - n^+)^{-1} = \exp\left[(c_0 + n_0 - 1)\, ax_1\right] / f_2^+(x_2).$$

The function f_2^+ is found from the boundary condition on the line $x_2 = x_1$

$$f_2^+ = -(1 - c_0)\exp\left[(c_0 + n_0 - 1)\, x_2\right] / n_0.$$

Thus, in the region D_1

$$n = n^+ = \frac{n_0(n_0 + c_0 - 1)}{n_0 - (1 - c_0)\exp\left[-a(c_0 + n_0 - 1)(x_1 - x_2)\right]}$$

$$c = c^+ = 1 - \frac{(1 - c_0)(c_0 + n_0 - 1)}{n_0 - (1 - c_0)\exp\left[-a(c_0 + n_0 - 1)(x_1 - x_2)\right]}.$$

The condition of continuity of the function n on the characteristic $x_2 = 0$ gives

$$n^+ = n^-, \quad x_2 = 0; \qquad \frac{\partial n^+}{\partial x_1} = \frac{\partial n^-}{\partial x_1}, \quad x_2 = 0.$$

Hence and from (2.3) there follows

$$f_1^- = f_1^+$$
$$n^-(c_0 + n_0 - 1 - n^-)^{-1} = \exp\left[(c_0 + n_0 - 1)\, ax_1\right] / f_2^-(x_2) \tag{2.4}$$

and the arbitrary function f_2^- must satisfy the condition

$$f_2^-(0) = f_2^+(0) = -(1 - c_0) / n_0 . \qquad (2.5)$$

Substituting (2.4) into the second equation of system (2.3) and using condition (2.2), we obtain the differential equation

$$df_2^- / dx_2 = -a(f_2^- + 1) ,$$

whose integral, with condition (2.5), will be

$$f_2^- = (c_0 + n_0 - 1) \exp(-ax_2) / n_0 - 1 .$$

Consequently, in region D_2

$$n = n^- = \frac{n_0 (c_0 + n_0 - 1)}{n_0 + [(c_0 + n_0 - 1) \exp(-ax_2) - n_0] \exp[-(c_0 + n_0 - 1) ax_1]}$$

$$c = c^- = n_0 \frac{1 - \exp[-(c_0 + n_0 - 1) ax_1]}{n_0 + [(c_0 + n_0 - 1) \exp(-ax_2) - n_0] \exp[-(c_0 + n_0 - 1) ax_1]} .$$

In conclusion, we present the solution of problem (2.1), (2.2) for the case $n_0 + c_0 = 1$:

$$\left.\begin{aligned} n &= n^+ = n_0 [an_0 (x_1 - x_2) + 1]^{-1} \\ c &= c^+ = 1 - n^+ \end{aligned}\right\} \quad (x_1, x_2) \in D_1$$

$$\left.\begin{aligned} n &= n^- = n_0 [n_0 ax_1 + \exp(-ax_2)]^{-1} \\ c &= c^- = an_0 x_1 [an_0 x_1 + \exp(-ax_2)]^{-1} \end{aligned}\right\} \quad (x_1, x_2) \in D_2 .$$

The graphs in Fig. 2 represent the distribution of salts in the solid phase (solid lines) and the concentration of ground water in the soil layer (dashed lines) for the cases $a = 1$, $\tau = 0.5, 1.0$, $n_0 = 2$, $c_0 = 0.2$.

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15 August 1968

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